# Integer programming tools for games

Press Start

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**CHAIRE FRQ-IVADO**

SCIENCE DES DONNÉES POUR LA THÉORIE DES JEUX COMBINATOIRES

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# Contents

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[Maschler et al., 2013]

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*"Game theory is the name given to the methodology of using mathematical tools to model and analyze situations of interactive decision making."*

Each player aims to **optimize** their utility.

So let us start with a recap of some **optimization results**.

Linear program

$$\min_{x} \sum_{i=1}^{n} c_i x_i = c^T x$$
$$s.t. \quad Ax \leq b$$
$$x \geq 0.$$

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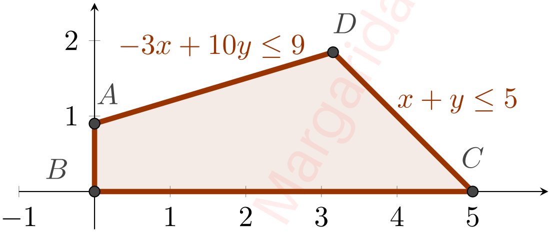

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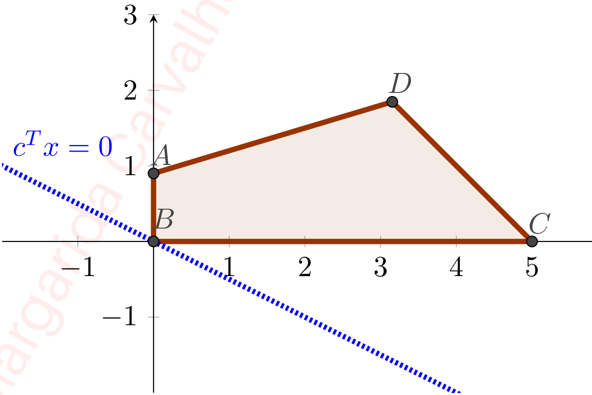




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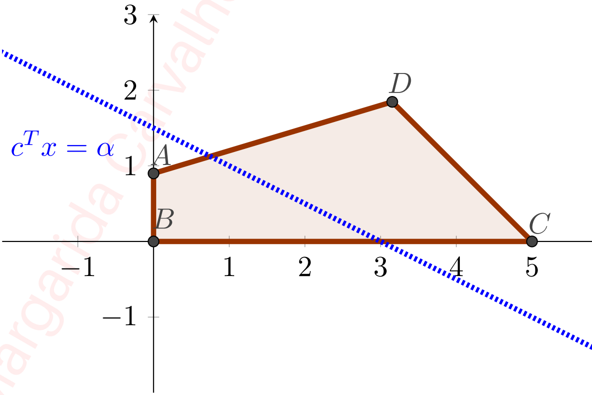




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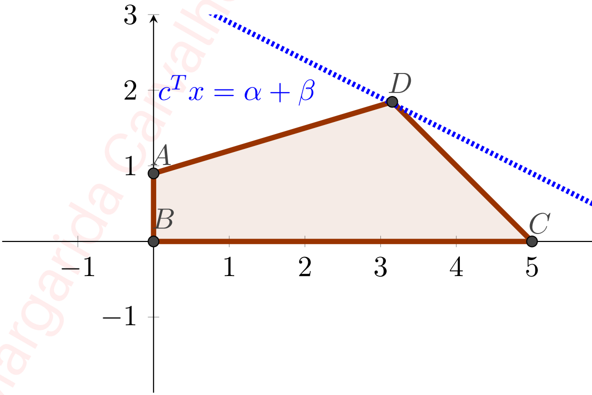




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# Representation of polyhedra

## Theorem

*Let $X \subseteq \mathbb{R}^n$ be a non-empty polyhedron with at least one extreme point. Let $x^1, \dots, x^k$ be the extreme points and $w^1, \dots, w^r$ be the extreme rays. Then*

$$X = \{x \in \mathbb{R}^n : x = \sum_{i=1}^{k} \lambda_i x^i + \sum_{j=1}^{r} \theta_j w^j, \sum_{i=1}^{k} \lambda_i = 1, \lambda_i \geq 0, \theta_j \geq 0\}$$

$$= Polytope + Cone \qquad \text{Minkowski sum}$$

$$= conv(\{x^1, \dots, x^k\}) + cone(\{w^1, \dots, w^r\})$$

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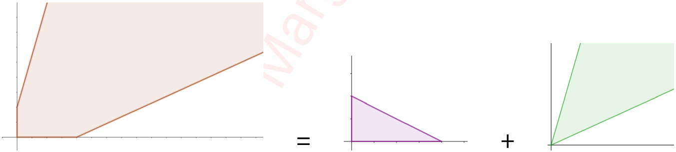

# Duality and complementary slackness



Primal problem

$$x^\star \in \arg\max_x \; c^T x$$
$$s.t. \quad Ax \leq b$$
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Dual problem

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**Strong duality:** the optimal objective values of the primal and dual are equal.

**Complementary slackness:**

$$(y^\star)^T (b - Ax^\star) = 0$$
$$(x^\star)^T (A^T y^\star - c) = 0$$

## Problem (P)

$$
\begin{aligned}
\min_{x} \quad & f(x) \\
s.t. \quad & g_i(x) \leq 0 \qquad i = 1, \ldots, m \\
& h_j(x) = 0 \qquad j = 1, \ldots, r
\end{aligned}
$$

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## Lagrangian

$$L(x, u, v) = f(x) + \sum_{i=1}^{m} u_i g_i(x) + \sum_{j=1}^{r} v_j h_j(x)$$

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## KKT conditions

$$\begin{aligned} & \nabla_x L = 0 & \\ & u_i \cdot g_i(x) = 0 & i = 1, \ldots, m \\ & g_i(x) \leq 0 & i = 1, \ldots, m \\ & h_j(x) = 0 & j = 1, \ldots, r \\ & u_i \geq 0 & i = 1 \ldots, m \end{aligned}$$

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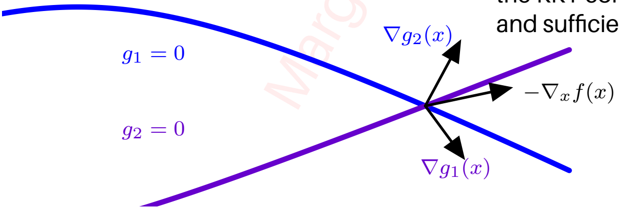
Under constraint qualification, any local minimum of (P) satisfies the KKT conditions. Under convexity, the KKT conditions are necessary and sufficient for global optimality.



## Definition (LCP [Cottle et al., 2009])

Given $q^n \in \mathbb{R}^n$ and $M \in \mathbb{R}^{n \times n}$, the linear complementarity problem, searches for $z \in \mathbb{R}^n$ such that

$$
\begin{aligned}
z &\geq 0 \\
q + Mz &\geq 0 \\
z^T(q + Mz) &= 0 \qquad \Leftrightarrow w = q + Mz, z^T w = 0, w \geq 0
\end{aligned}
$$

The theory of LCPs is particularly useful for bimatrix games and continuous games (with concave problems for each player).

$$z \geq 0, \quad q + Mz \geq 0, \quad z^T(q + Mz) = 0$$

## Player X

$$\min_x \; {c^T}^X x + x \cdot C^X \cdot y + \frac{1}{2} x^T Q^X x$$
$$s.t. \quad Ax \geq b$$
$$x \geq 0$$

## KKT conditions

$$\alpha = c^X + C^X y + Q^X x - A^T \mu$$
$$\nu = -b + Ax$$
$$x^T \alpha = 0$$
$$\mu^T \nu = 0$$
$$x \geq 0, \mu \geq 0, \alpha \geq 0, \nu \geq 0$$

## Player Y

$$\min_y \; {c^T}^Y y + y \cdot C^Y \cdot x + \frac{1}{2} y^T Q^Y y$$
$$s.t. \quad Dy \geq f$$
$$y \geq 0$$

## KKT conditions

$$\beta = c^Y + C^Y x + Q^Y y - D^T \lambda$$
$$\eta = -f + Dy$$
$$y^T \beta = 0$$
$$\lambda^T \eta = 0$$
$$y \geq 0, \lambda \geq 0, \beta \geq 0, \eta \geq 0$$

$$z \geq 0, \quad q + Mz \geq 0, \quad z^T(q + Mz) = 0$$

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$$y \geq 0, \lambda \geq 0, \beta \geq 0, \eta \geq 0$$

$$q = \begin{bmatrix} c^X \\ -b \\ c^Y \\ -f \end{bmatrix} \qquad M = \begin{bmatrix} Q^X & -A^T & C^X & 0 \\ A & 0 & 0 & 0 \\ C^Y & 0 & Q^Y & -D^T \\ 0 & & D & 0 \end{bmatrix} \qquad z = \begin{bmatrix} x \\ \mu \\ y \\ \lambda \end{bmatrix}$$

# LCPs and integer programming

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$$\Leftrightarrow \bigwedge_i (z_i \leq Lx_i \wedge (q + Mz)_i \leq L(1 - x_i) \wedge x_i \in \{0, 1\}) \wedge z_i \geq 0 \quad \wedge \quad (q + Mz)_i \geq 0$$

$L$ sufficiently large

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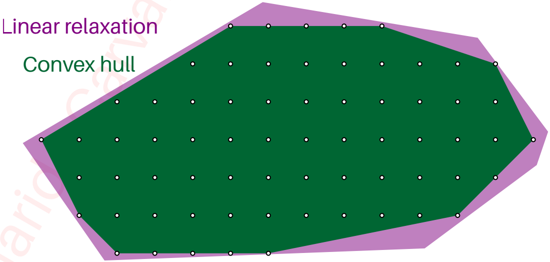

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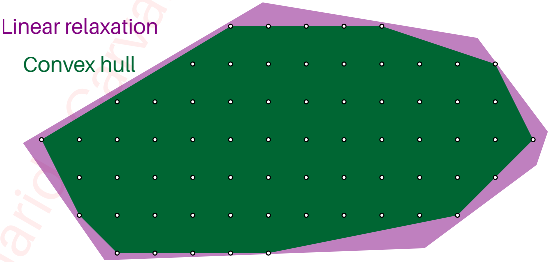


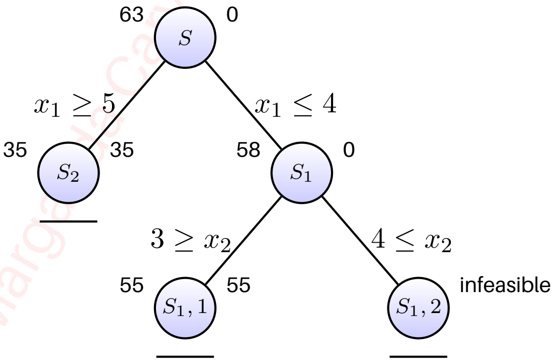

Ideally, we'd like to know the convex hull of the feasible region of (IP).

# Branch-and-bound by Ailsa Land and Alison Doig

Integer program (IP)

$$\max_{x} \; 7x_1 + 9x_2$$
$$s.t. \quad -x_1 + 3x_2 \leq 6$$
$$7x_1 + x_2 \leq 35$$
$$x_2 \leq 7$$
$$x_1, x_2 \in \mathbb{Z}_+.$$

$(0, 0)$ is feasible and thus, we have the lower bound $0$.



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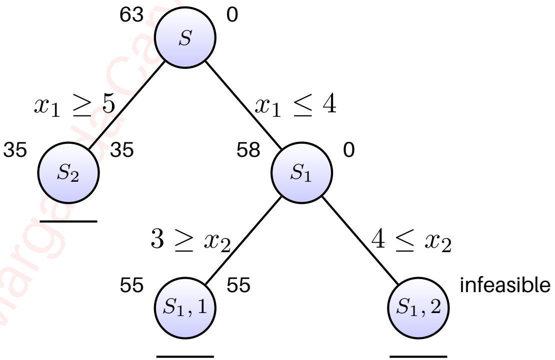
$S : (x_1, x_2) = (4.5, 3.5)$



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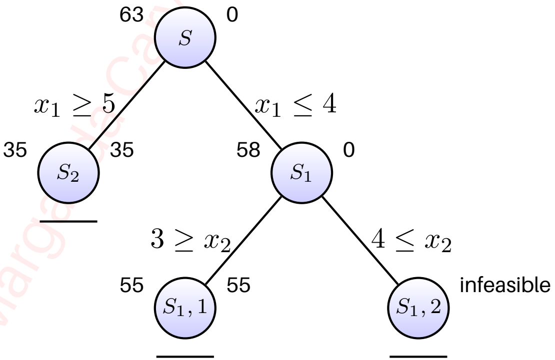
$S_1 : (x_1, x_2) = (4, \frac{10}{3})$



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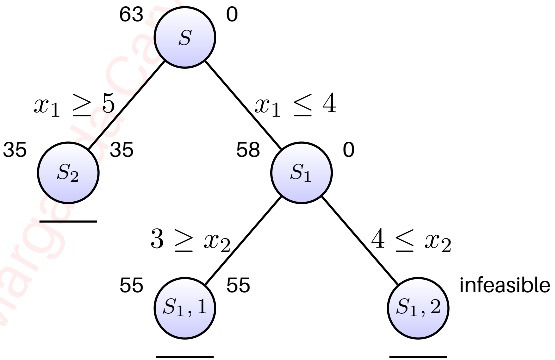

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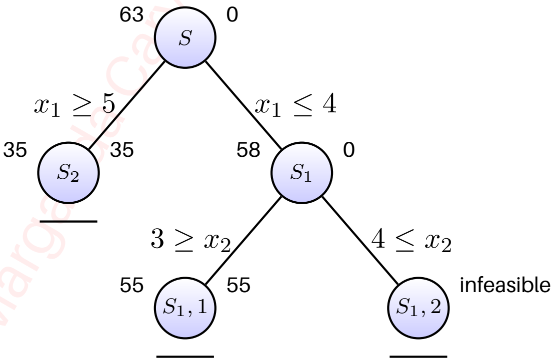
$S_1, 2$ : infeasible



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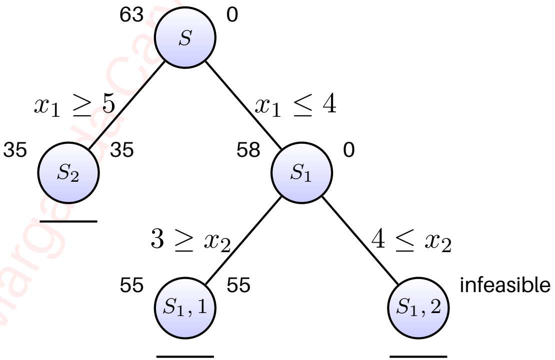
$S_2 : (x_1, x_2) = (5, 0)$



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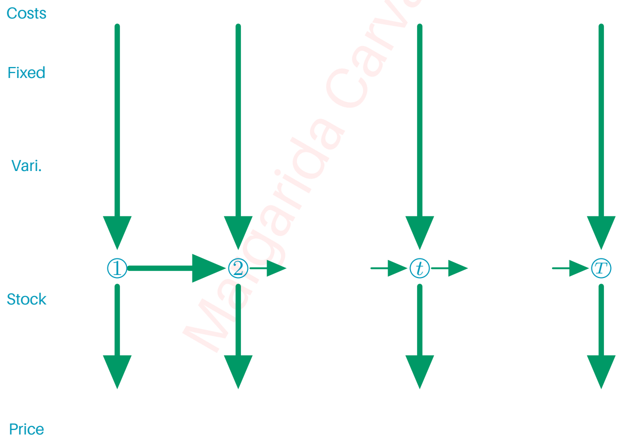
- Although the **theoretical intractability** of IPs, we have powerful tools that can solve them efficiently in practice.
- Next, we will see some examples of non-convex games which can benefit from IP tools.

# Lot-sizing game



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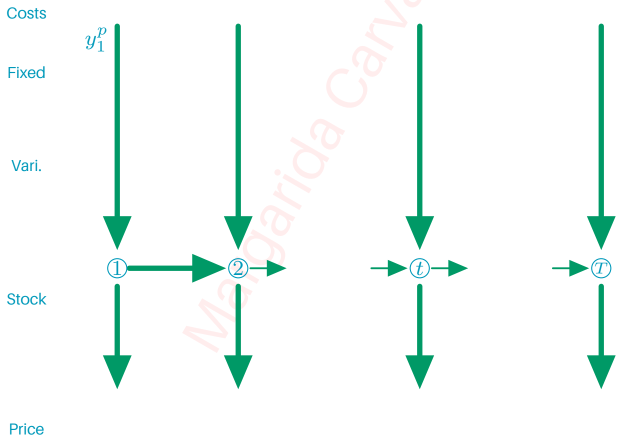

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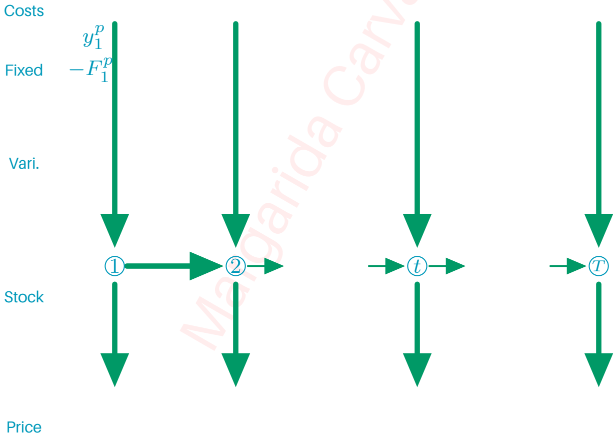

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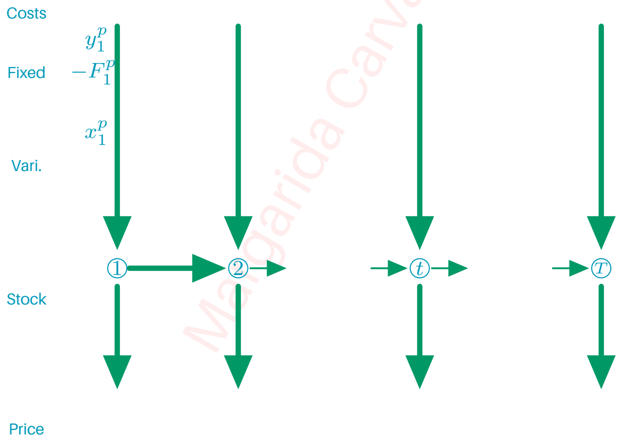

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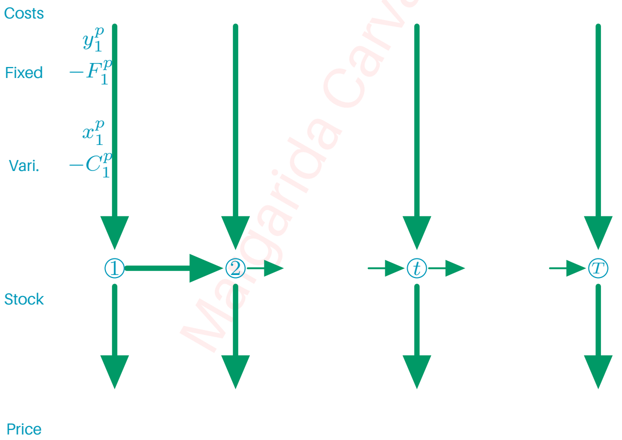

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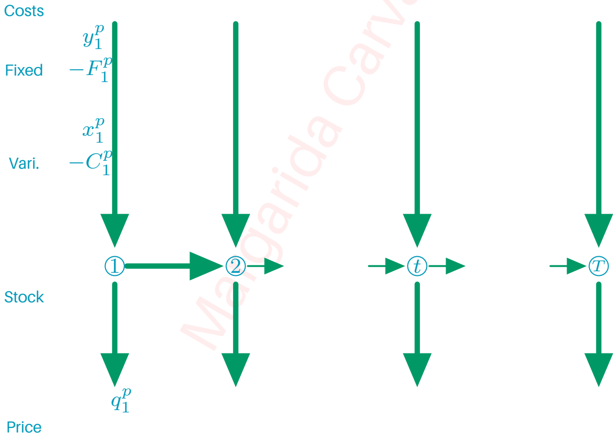

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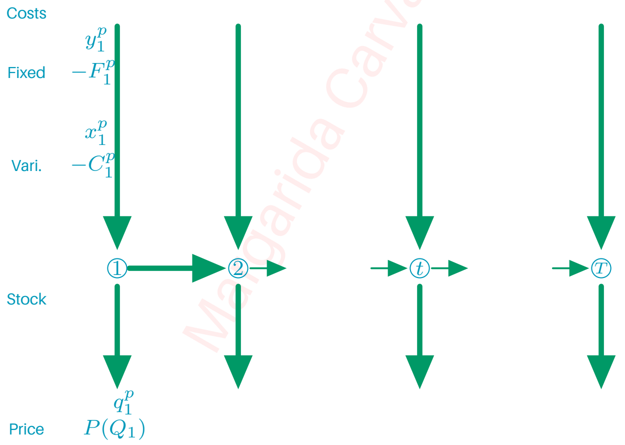

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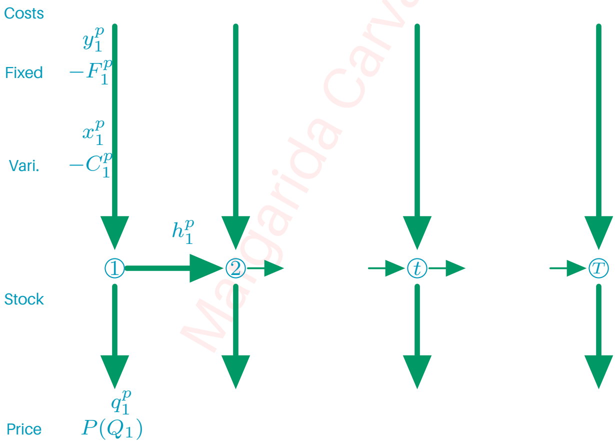

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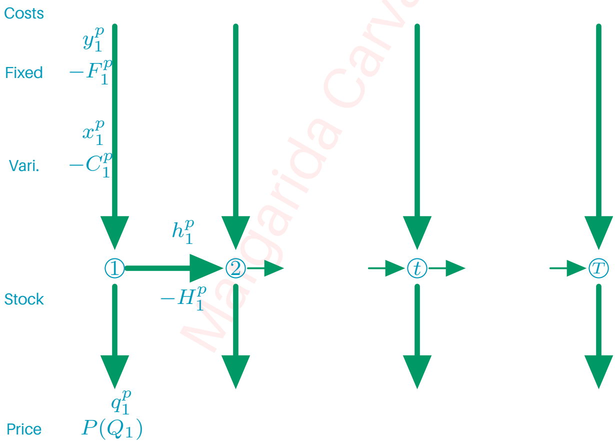

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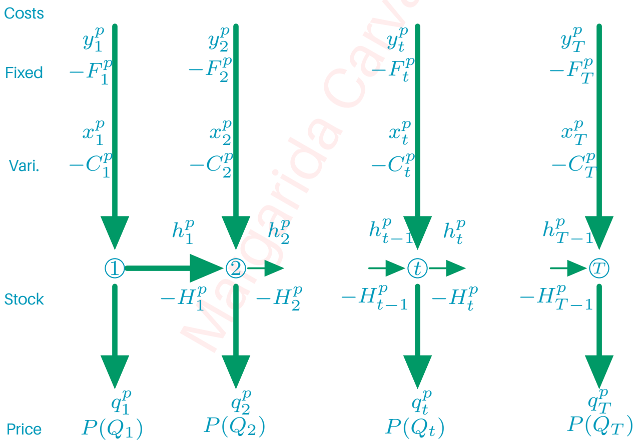

# Lot-sizing game [Carvalho et al., 2018]

Each player $i = 1, 2, \ldots, m$ solves the following parametric mathematical programming problem

$$
\begin{aligned}
\max_{y^i, x^i, q^i, h^i} \quad & \sum_{t=1}^{T} P(Q_t) q_t^i - \sum_{t=1}^{T} F_t^i y_t^i - \sum_{t=1}^{T} H_t^i h_t^i - \sum_{t=1}^{T} C_t^i x_t^i \\
\text{subject to} \quad & x_t^i + h_{t-1}^i = h_t^i + q_t^i \qquad \text{for } t = 1, \ldots, T \\
& 0 \leq x_t^i \leq M y_t^i \qquad \text{for } t = 1, \ldots, T \\
& h_0^i = h_T^i = 0 \\
& y_t^i \in \{0, 1\} \qquad \text{for } t = 1, \ldots, T
\end{aligned}
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# Kidney exchange game



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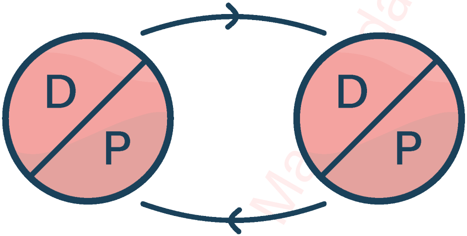

# Kidney exchange game

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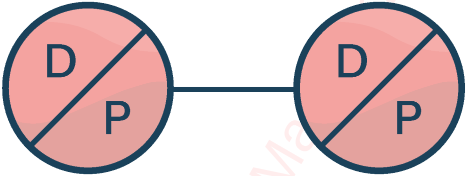

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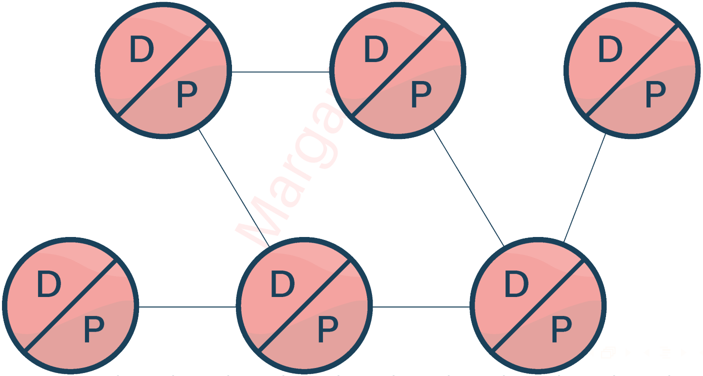

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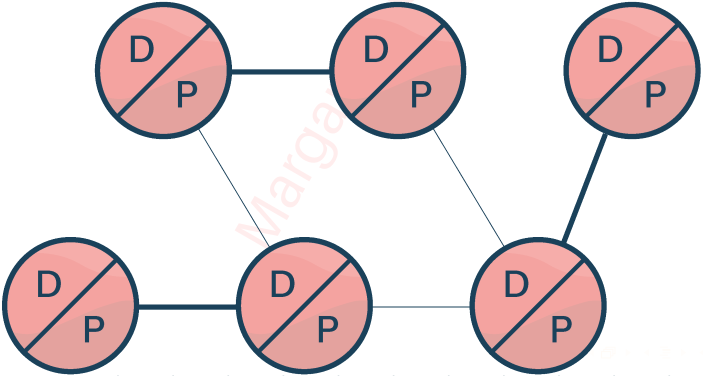


# Kidney exchange game

- Many national KEPs in Europe are the result of collaboration between transplant centers;
- Some centers run their own program in parallel to the national KEPs;
- Nowadays, many countries have national KEPs.

# Kidney exchange game

- Many national KEPs in Europe are the result of collaboration between transplant centers;
- Some centers run their own program in parallel to the national KEPs;
- Nowadays, many countries have national KEPs.

This motivates a game theoretical analysis.

Literature: [Roth et al., 2005, Sönmez and Ünver, 2013, Ashlagi and Roth, 2011, Toulis and Parkes, 2011, Ashlagi et al., 2015, Caragiannis et al., 2015, Ashlagi and Roth, 2014, Toulis and Parkes, 2015, Blum et al., 2017, Klimentova et al., 2020, Biró et al., 2020]

# Kidney exchange game [Carvalho et al., 2017, Carvalho and Lodi, 2022]

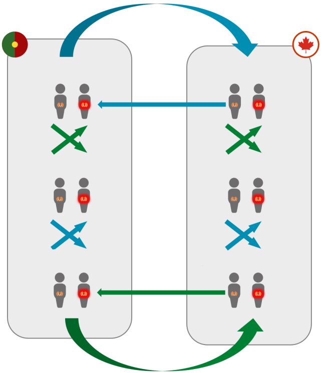
## Players

- **Player A** controls the incompatible patient-donor vertices
- **Player B** controls the incompatible patient-donor vertices

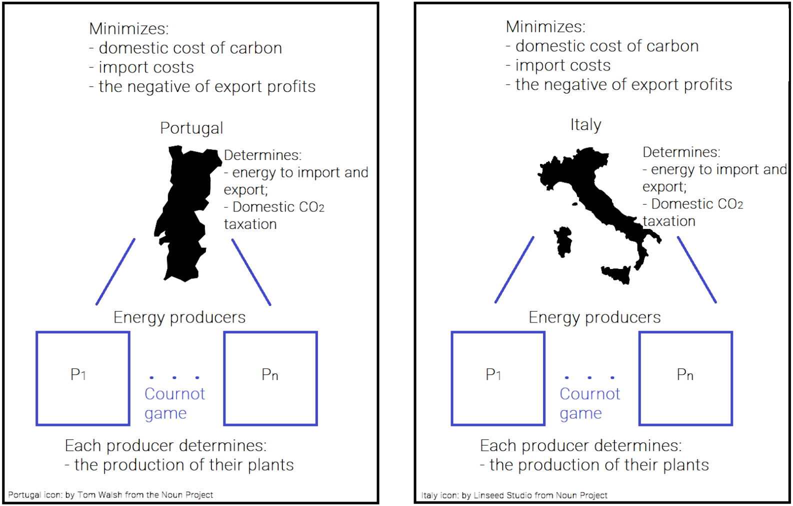
Country A solves the following parametric mathematical program

$$\max_{x^A \in \{0,1\}^{|C^A|+|I|}} \sum_{c \in C^A} w_c^A x_c^A + \sum_{c \in I} w_c^A x_c^A x_c^B$$

$$\text{s. t.} \quad \sum_{c \in C^A : i \in c} x_c^A + \sum_{c \in I : i \in c} x_c^A \leq 1 \quad \forall i \in V^A$$



# Energy trade game [Carvalho et al., 2021a]



# Mathematical programming game [Dragotto et al., 2021, Dragotto, 2022]

## Definition

A *mathematical programming game* (MPG) is a game among $n$ players with each player $p = 1, 2, \ldots, n$ solving the optimization problem

$$\max_{x \in X^p} \Pi^p(x^p, x^{-p})$$

where $X^p$ is the set of feasible solutions for player $p$.

The goal of each player in the game can be described through a parametric *mathematical program*.

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- What is the interest of solving MPGs?
- What is a solution for an MPG?
- What is the computational complexity of determining such solution?
- Can we extend the tools of (tractable) non-convex optimization to (some) MPGs?

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In this tutorial, we will focus on NASPs.

# Contents

# Motivation

1952 H. Stackelberg publishes *The theory of market economy*: a player, called the leader, takes his decision before decisions of other players, called the followers;

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1952 H. Stackelberg publishes *The theory of market economy*: a player, called the leader, takes his decision before decisions of other players, called the followers;

80's Understanding of the fundamental concepts;
Development policy (e.g. determination of pricing policies);
Generalization: multilevel programming - Hierarchical structures;
Computational complexity theory;

# Motivation



90's Algorithms to linear bilevel programming problems;

Algorithms to integer linear bilevel programming problems;

# Motivation



| | |
|---|---|
| 90's | Algorithms to linear bilevel programming problems;<br>Algorithms to integer linear bilevel programming problems; |
| Recently | Bilevel problem specific algorithms/heuristics;<br>Defence-planning problems (e.g. Transmission networks);<br>Worst-case analyses;<br>Interdiction problems (e.g. sensitivity analysis); |

# Reference

Continuously being updated:

[Vicente, L. N. and Calamai, P. H., 1994]
Bilevel and multilevel programming: A bibliography review.

# Definition

Bilevel Programming Problem (BPP):

$$\text{Minimize}_{x,y} \; L(x,y)$$

$$\text{subject to} \quad (x,y) \in X$$

where $y$ solves the follower's problem

$$\text{Minimize}_y \; F(x,y) \quad \text{s.t.} \quad (x,y) \in Y$$

# Definition



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Relaxed feasible set

$$\Omega = \{(x,y) : (x,y) \in X \text{ and } (x,y) \in Y\}$$

# Definition



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Lower level feasible set

$$\Omega(x) = \{y : (x,y) \in Y\}$$

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Follower's best reaction to $x$ is the set

$$M(x) = \{y \in \arg\min\{F(x,y') : y' \in \Omega(x)\}\}$$

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Induced region is the set (bilevel feasibility)

$$IR = \{(x,y) : (x,y) \in \Omega \text{ and } y \in M(x)\}$$

which in general is non-convex.

# Definition

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Don't mix with bi-objective optimization problems.

# Challenges

- The problem may not be well-defined.

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- The problem may not be well-defined.
- Even the linear BPP is NP-hard.

# Example: optimistic and pessimistic cases



Consider the following Stackelberg Competition:

$$\max_{x^A} \left(20 - (x^A + x^B)\right) x^A - 10x^A$$

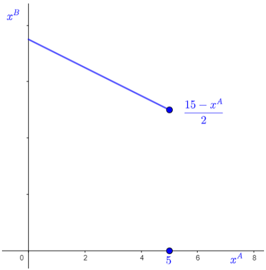
s. t. $x^A \geq 0$

where $x^B$ solves the follower's problem

$$\max_{x^B, y^B} \left(20 - (x^A + x^B)\right) x^B - 5x^B - 25y^B$$

s. t. $0 \leq x^B \leq My^B$

$y^B \in \{0, 1\}.$




If $x^A = 5$, then $x^B(5) = 5$ or $x^B(5) = 0$, and the leader's profit is 0 or 25, respectively.

Given a strategy of the leader, the follower may have multiple optimal strategies.

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Furthermore, these equivalent follower's strategies can yield different objective values for the leader.

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**Happy Leader**

In the *optimistic* scenario , the follower picks the strategy that yields the **best objective value** for the leader.

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Furthermore, these equivalent follower's strategies can yield different objective values for the leader.

**Happy Leader**
In the *optimistic* scenario , the follower picks the strategy that yields the **best objective value** for the leader.

**Sad Leader**
In the *pessimistic* scenario, the follower picks the strategy that yields the **worst objective value** for the leader.

# Example

## Going back to the Stackelberg Competition:

$$\max_{x^A} \left(20 - (x^A + x^B)\right) x^A - 10x^A$$

s. t. $x^A \geq 0$

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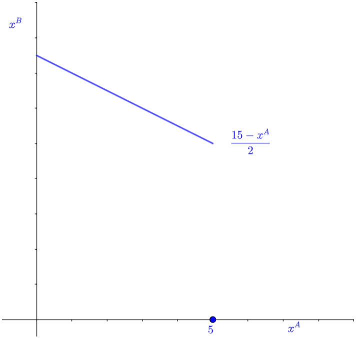


**Optimistic formulation**

The optimal solution is $(x^A, x^B, y^B) = (5, 0, 0)$

# Example

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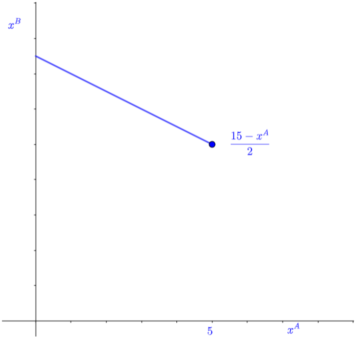


**Pessimistic formulation**

The problem feasible region is a non-compact set which (in this case) leads to the non-existence of an equilibrium. The leader has incentive to choose $x^A = 5 + \epsilon$ with $\epsilon > 0$ very small.

How do we solve BPP?

- Convex BPPs
- Non-convex BPPs (Integer BPPs)

# Convex BPPs

$$\text{Minimize}_{x,y}\ L(x,y)$$

subject to $(x,y) \in X$

where $y$ solves the follower's problem

$$\text{Minimize}_{y}\ F(x,y) \quad \text{s.t. } (x,y) \in Y$$

# Convex BPPs

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$F(x,y)$ is a convex function and $Y$ is a convex set in $y$ for all values of $x$.

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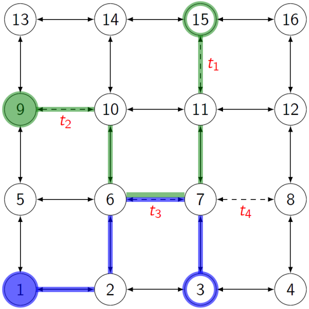
$F(x,y)$ is a convex function and $Y$ is a convex set in $y$ for all values of $x$. The lower level problem can be replaced by its KKT-conditions to obtain an equivalent single-level problem.

# Example: Network pricing problem [Labbé et al., 1998, Bui et al., 2022]






- Leader prices $t_1, t_2, t_3, t_4$.
- Follower 1 goes from 1 to 3.
  Follower 2 foes from 9 to 15.

# Example: Network pricing problem [Labbé et al., 1998, Bui et al., 2022]



$$\max_{t \geq 0, x, y} \sum_{k \in K} \eta^k t_a x_a^k$$

$$\forall k \in K \begin{cases} (x^k, y^k) \in \arg\min_{\hat{x}, \hat{y}} \sum_{a \in A_1} (c_a + t_a)\hat{x}_a^k + \sum_{a \in A_2} c_a \hat{y}_a \\ \sum_{a \in A_1^+(i)} \hat{x}_a + \sum_{a \in A_2^+(i)} \hat{y}_a - \sum_{a \in A_1^-(i)} \hat{x}_a + \sum_{a \in A_2^-(i)} \hat{y}_a = b_i^k, \quad i \in V, \\ \hat{x}_a \in \{0, 1\}, \quad a \in A_1, \\ \hat{y}_a \in \{0, 1\}, \quad a \in A_2, \end{cases}$$

where $b_i^k = 1$ if $i = o^k$, $-1$ if $i = d^k$, and $0$ otherwise.

# Example: Network pricing problem [Labbé et al., 1998, Bui et al., 2022]

$$\max_{t\geq 0, x} \sum_{k \in K} \eta^k t^T x^k$$

$$\forall k \in K \begin{cases} x^k \in \arg\min_{\hat{x}^k} (c+t)^T \hat{x}^k \\ \qquad A\hat{x}^k = b^k \\ \qquad \hat{x}^k \geq 0 \end{cases}$$

Dual

$$\forall k \in K \begin{cases} y^k \in \arg\max_{\hat{y}^k} (b^k)^T \hat{y}^k \\ \qquad A^T \hat{y}^k \leq c + t \end{cases}$$

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Standard formulation: strong duality

$$\max_{t \geq 0, x, y} \sum_{k \in K} \eta^k t^T x^k$$

$$\forall k \in K \begin{cases} Ax^k = b^k \\ x^k \geq 0 \\ A^T y^k \leq c + t \\ (c+t)^T x^k = (b^k)^T y^k \end{cases}$$

# Example: Network pricing problem [Labbé et al., 1998, Bui et al., 2022]

$$\max_{t\geq 0, x} \sum_{k\in K} \eta^k t^T x^k$$

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Dual

$$\forall k \in K \begin{cases} y^k \in \arg\max_{\hat{y}^k} (b^k)^T \hat{y}^k \\ A^T \hat{y}^k \leq c+t \end{cases}$$

Standard formulation: complementary slackness

$$\max_{t\geq 0, x, y} \sum_{k\in K} \eta^k t^T x^k$$

$$\forall k \in K \begin{cases} Ax^k = b^k \\ x^k \geq 0 \\ A^T y^k \leq c+t \\ ((c+t)^T - A^T y^k) x^k = 0 \end{cases}$$

# Integer BPPs - Example from Moore and Bard (1990)

$$\min_{x,y} \; -x - 10y$$

s.t. $x \in P_U$

$$\min_{y} \; y$$

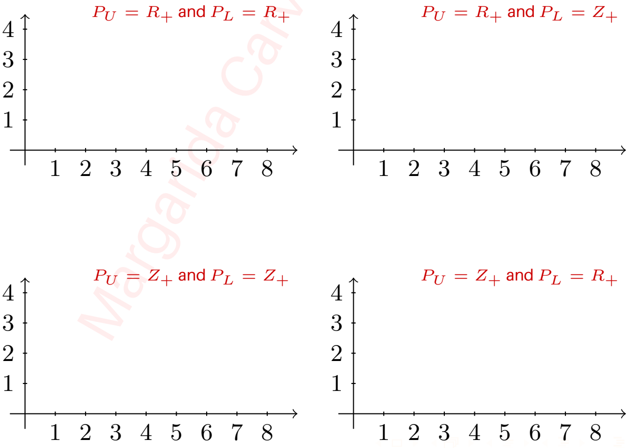
s.t $5x - 4y \geq -6$

$-x - 2y \geq -10$

$-2x + y \geq -15$

$2x + 10y \geq 15$

$y \in P_L$



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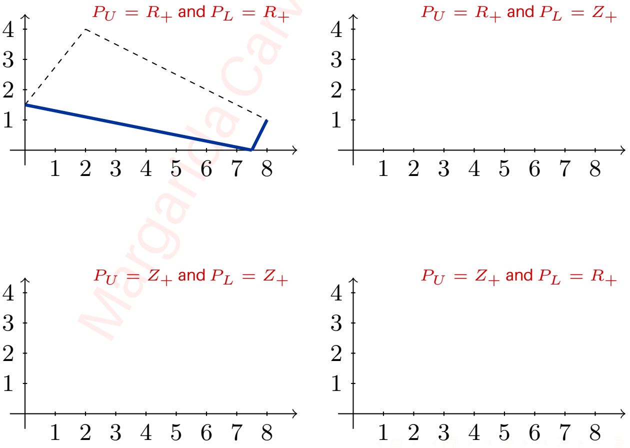

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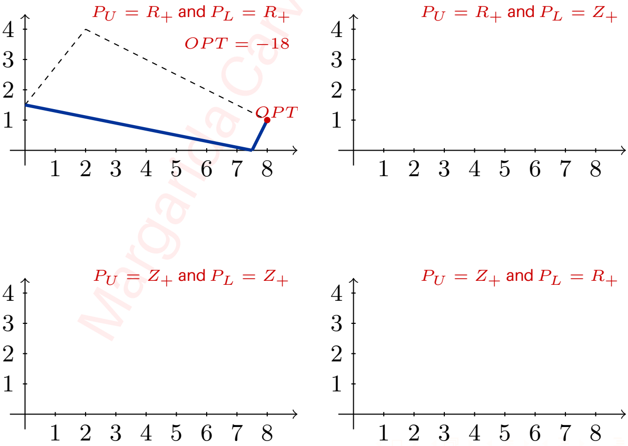

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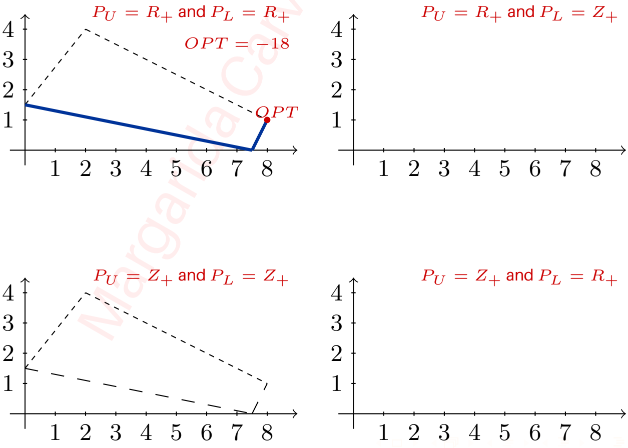

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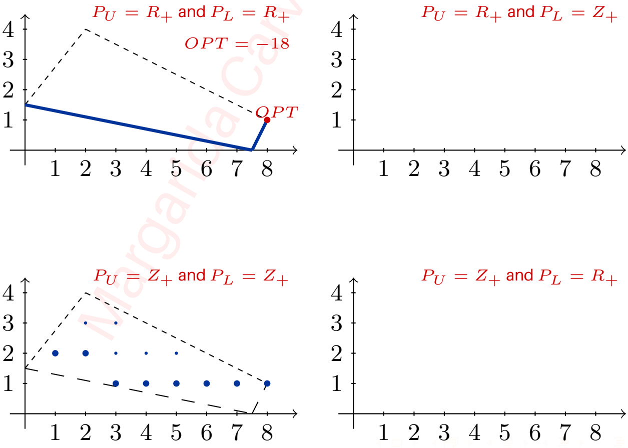

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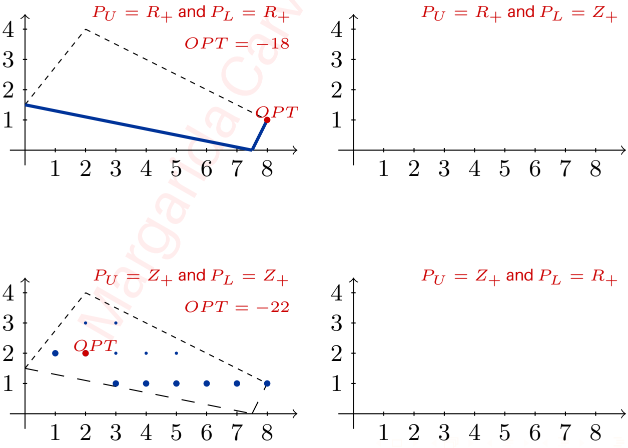

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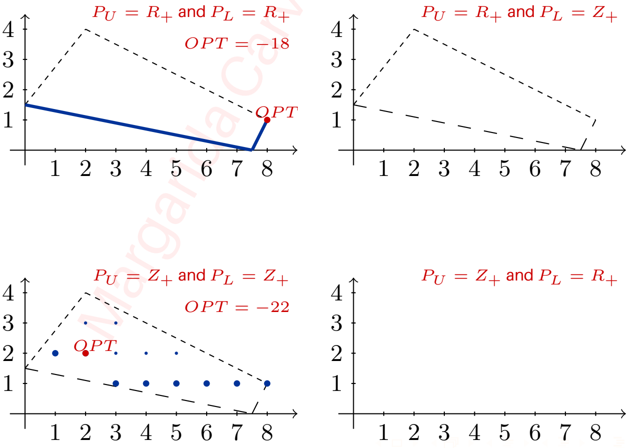

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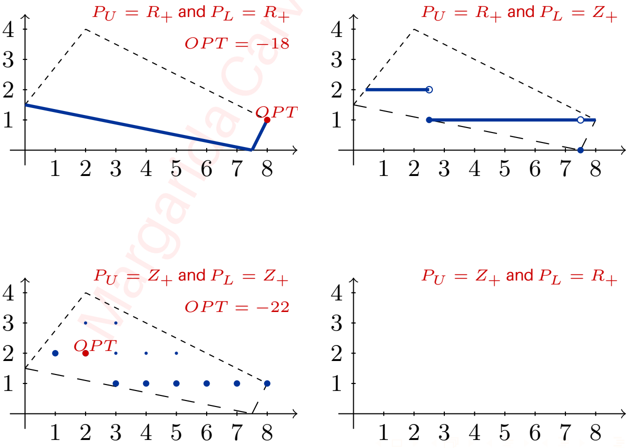

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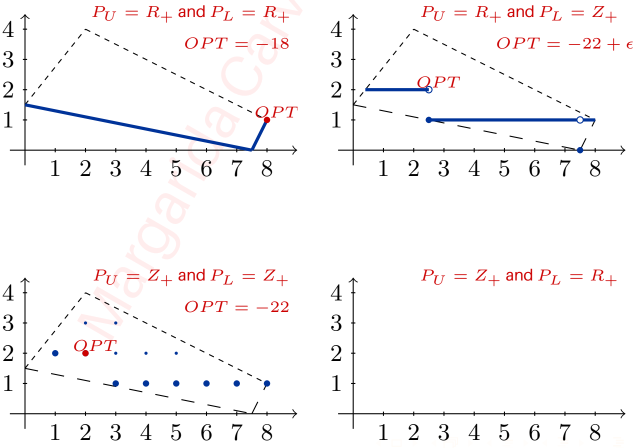

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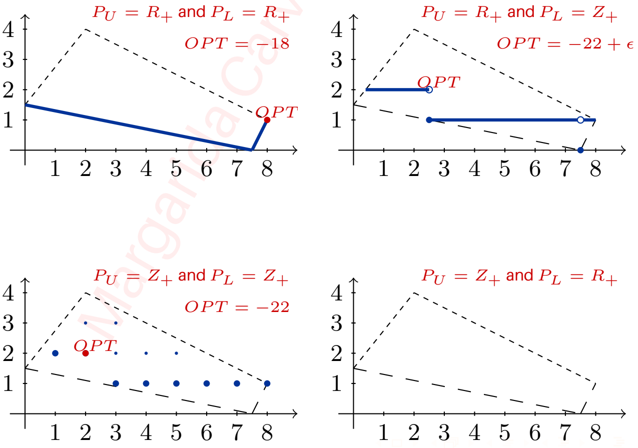

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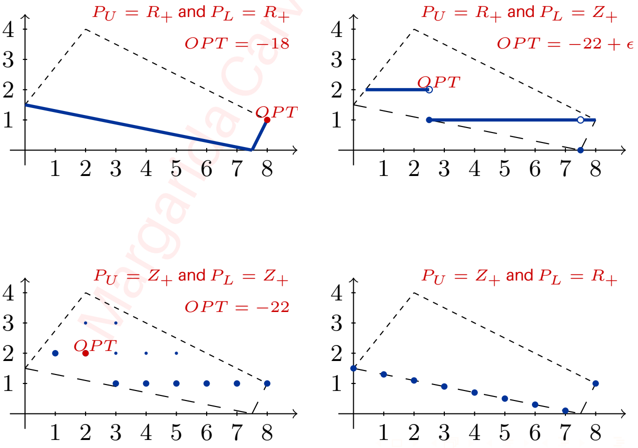

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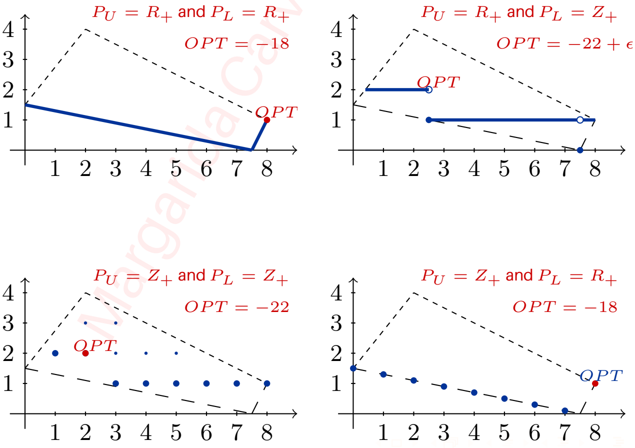

# Branch-and-Bound



Minimize$_{x,y}$ $L(x, y)$

subject to $(x, y) \in X$

$x_i$ integer for some $i$

where $y$ solves the follower's problem

Minimize$_y$ $F(x, y)$

s.t. $(x, y) \in Y$

$y_i$ integer for some $i$

# Branch-and-Bound

Relax integrality and drop follower's objective

Minimize$_{x,y}$ $L(x, y)$

subject to $(x, y) \in X$

~~$x_i$ integer for some $i$~~

where $y$ solves the follower's problem

~~Minimize$_y$ $F(x, y)$~~

s.t. $(x, y) \in Y$

~~$y_i$ integer for some $i$~~

# Branch-and-Bound

High-point problem

$$\begin{aligned} \text{Minimize}_{x,y}\ & L(x,y) \\ \text{subject to}\ & (x,y) \in X \\ & (x,y) \in Y \end{aligned}$$

It is a **lower bound.**

# Branch-and-Bound



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It is a **lower bound**.
Solve optimization problem.
**If** the solution is fractional in some integer variables, branch.

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**Else**, verify if it is bilevel feasible.

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High-point problem

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It is a **lower bound**.
Solve optimization problem.
**If** the solution is fractional in some integer variables, branch.
**Else**, verify if it is bilevel feasible. **If** it is bilevel feasible update the incumbent.
**Else** branch or cut the solution.

# Literature: Branch-and-Cut

Cuts for mixed integer bilevel programming:

[S. DeNegre, 2011]
Interdiction and discrete bilevel linear programming, Ph.D. thesis, Lehigh University

[M. Fischetti, I. Ljubic, M. Monaci, and M. Sinnl, 2018]
On the use of intersection cuts for bilevel optimization. Mathematical Programming

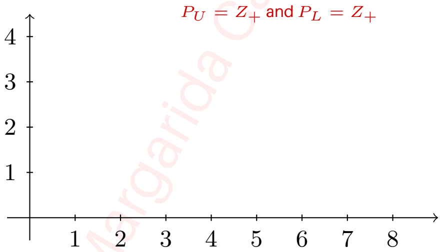
[S. Tahernejad, T. Ralphs, S. DeNegre, 2020]
A branch-and-cut algorithm for mixed integer bilevel linear optimization problems and its implementation. Mathematical Programming Computation

[K. Tanınmış, M. Sinnl, 2022]
A Branch-and-Cut Algorithm for Submodular Interdiction Games. INFORMS Journal on Computing

Currently, the solver **MibS** by Ted Ralphs is available and **Bilevel Integer Programming Solver** by Markus Sinnl et al.

# Literature: Branch-and-Cut

Intersection cuts idea:




Make your intersection cut!

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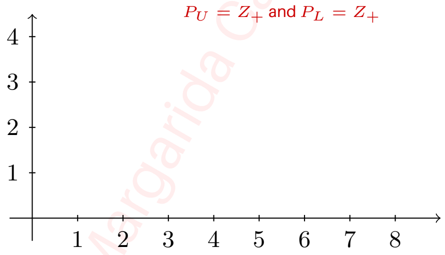


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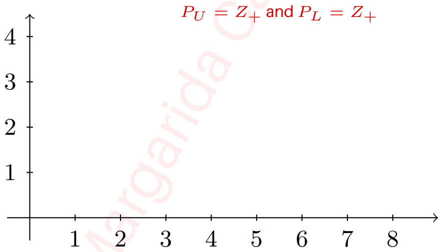


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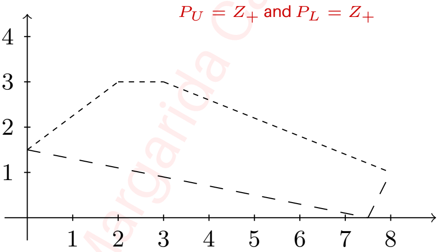


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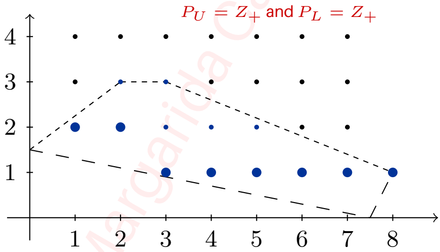


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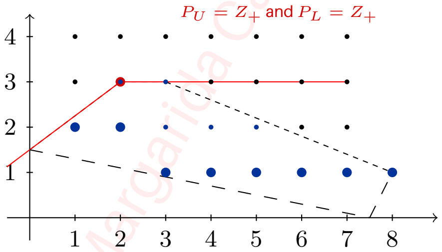


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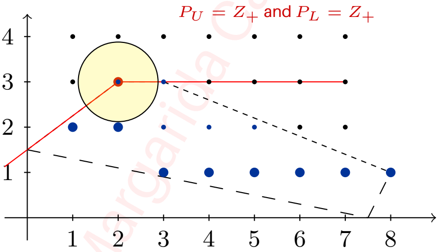


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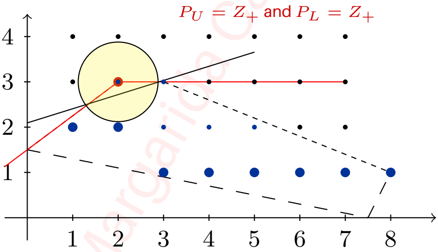



Make your intersection cut!

# Contents

# Stackelberg game

Latin leader

$$\begin{aligned} \min_{x,y} \quad &: \quad c^T x + d^T y \\ \text{subject to} \quad & Ax + By \leq b \\ & y \in \arg\min_y \left\{ f^T y : Qy \leq g - Px \right\} \end{aligned}$$

# Stackelberg game

Trivial NASP

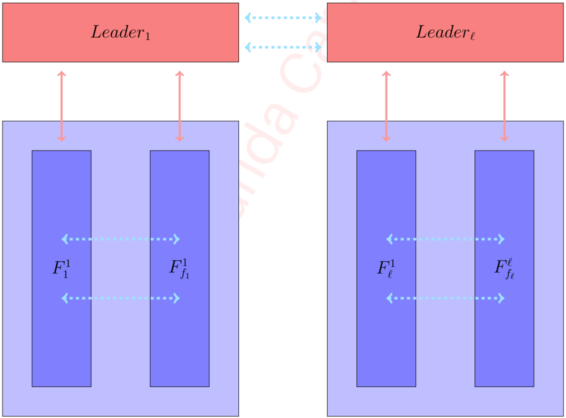
Latin leader

$$\begin{aligned}\min_{x,y} \quad &: \quad c^T x + d^T y + \left(G\begin{pmatrix}\xi\\ \chi\end{pmatrix}\right)^T\begin{pmatrix}x\\ y\end{pmatrix}\\ \text{subject to} \quad & \quad Ax + By \quad \leq \quad b\\ & \quad y \quad \in \quad \arg\min_{y}\left\{f^T y : Qy \leq g - Px\right\}\end{aligned}$$

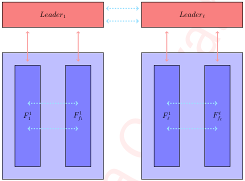
Greek leader

$$\begin{aligned}\min_{\xi,\chi} \quad &: \quad \alpha^T \xi + \beta^T \chi + \left(\Gamma\begin{pmatrix}x\\ y\end{pmatrix}\right)^T\begin{pmatrix}\xi\\ \chi\end{pmatrix}\\ \text{subject to} \quad & \quad \Phi\xi + \Psi\chi \quad \leq \quad \rho\\ & \quad \chi \quad \in \quad \arg\min_{\chi}\left\{\phi^T \chi : \Omega\phi \leq \gamma - \Pi\xi\right\}.\end{aligned}$$

# Nash Games among Stackelberg Players (Leaders) [Carvalho et al., 2021a]



# NASP




## Definition (NASP)

A *NASP* is a linear Nash game $N = (P^1, \dots, P^k)$ where for each $i$, $P^i$ is a simple Stackelberg game:

$$P^i \qquad \min_{x^i \in \mathcal{R}^{n_i}} \{f^i(x^i; x^{-i}) : x^i = (z^i, y^i) \in \mathcal{F}_i, y^i \in SOL(P(z^i))\}$$

$f^i$ is linear
$\mathcal{F}_i$ is a polyhedron
$SOL(P(z^i))$ is the set of Nash equilibria for the game played by the followers
Followers have quadratic convex objectives and polyhedral feasible regions.

### Theorem

*It is $\Sigma_2^p$-hard to decide if a trivial NASP instance has a pure Nash equilibrium, even if strategy sets are bounded.*

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### Corollary

*If each player's strategy set in a trivial* NASP *is a bounded set, an equilibrium exists.*

### Theorem

*It is $\Sigma_2^p$-hard to decide if a trivial* NASP *has an equilibrium.*

# Preliminaries

## Definition (LCP)

Given $q \in \mathbb{R}^n$ and $M \in \mathbb{R}^{n \times n}$, the linear complementarity problem, searches for $z \in \mathbb{R}^n$ such that

$$\begin{aligned} z &\geq 0 \\ q + Mz &\geq 0 \\ z^T(q + Mz) &= 0 \qquad \Leftrightarrow w = q + Mz, z^T w = 0, w \geq 0 \end{aligned}$$

The theory of LCPs is particularly useful for bimatrix games and continuous games (with concave problems for each player).

$$z \geq 0, \quad q + Mz \geq 0, \quad z^T(q + Mz) = 0$$

**Player X**

$$\min_x \; c^{T^X} x + x \cdot C^X \cdot y + \frac{1}{2} x^T Q^X x$$
$$s.t. \quad Ax \geq b$$
$$x \geq 0$$

**KKT conditions**

$$\alpha = c^X + C^X y + Q^X x - A^T \mu$$
$$\nu = -b + Ax$$
$$x^T \alpha = 0$$
$$\mu^T \nu = 0$$
$$x \geq 0, \mu \geq 0, \alpha \geq 0, \nu \geq 0$$

**Player Y**

$$\min_y \; c^{T^Y} y + y \cdot C^Y \cdot x + \frac{1}{2} y^T Q^Y y$$
$$s.t. \quad Dy \geq f$$
$$y \geq 0$$

**KKT conditions**

$$\beta = c^Y + C^Y x + Q^Y y - D^T \lambda$$
$$\eta = -f + Dy$$
$$y^T \beta = 0$$
$$\lambda^T \eta = 0$$
$$y \geq 0, \lambda \geq 0, \beta \geq 0, \eta \geq 0$$

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$$\begin{aligned} \min_x \ & {c^T}^X x + x \cdot C^X \cdot y + \frac{1}{2} x^T Q^X x \\ s.t. \ & Ax \geq b \\ & x \geq 0 \end{aligned}$$

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$$q = \begin{bmatrix} c^X \\ -b \\ c^Y \\ -f \end{bmatrix} \qquad M = \begin{bmatrix} Q^X & -A^T & C^X & 0 \\ A & 0 & 0 & 0 \\ C^Y & 0 & Q^Y & -D^T \\ 0 & D & 0 & 0 \end{bmatrix} \qquad z = \begin{bmatrix} x \\ \mu \\ y \\ \lambda \end{bmatrix}$$

# Preliminaries

## Theorem ([Cottle et al., 2009])

*Let $P$ be a facile Nash game. Then, there exist $M, q$ such that every solution to the LCP defined by $M, q$ is a pure Nash equilibrium for $P$ and every pure Nash equilibrium of $P$ solves the LCP.*

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Idea 1: The followers play a facile Nash game. We can find a pure equilibrium for it by solving an LCP.

$$y^i \in SOL(P(z^i)) \underbrace{\Leftrightarrow}_{KKT} 0 \leq (x^i, \lambda^i) \perp Mx^i + N\lambda^i + q \geq 0$$

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We will also show that the leader's problem can be transformed in a facile Nash game.

# Preliminaries

## Theorem ([Basu et al., 2021])

*Let $S$ be the feasible set of a simple Stackelberg game. Then, $S$ is a finite union of polyhedra. Conversely, let $S$ be a finite union of polyhedra. Then, there exists a simple Stackelberg game with $P(x)$ containing exactly $1$ player such that the feasible region of the simple Stackelberg game provides an extended formulation of $S$.*

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Idea 2: The followers' game can be replaced by a union of polyhedra.

$$(P^i) \quad \min_{x^i \in \mathcal{R}^{n_i}} \{f^i(x^i; x^{-i}) : x^i = (z^i, y^i) \in \mathfrak{F}_i, y^i \in SOL(P(z^i))\}$$

$$\Leftrightarrow \min_{x^i \in \mathcal{R}^{n_i}} \{f^i(x^i; x^{-i}) : x^i = (z^i, y^i) \in \mathfrak{F}_i, 0 \leq w^i = (x^i, \lambda^i) \perp M' w^i + q \geq 0\}$$

$L$ sufficiently large.

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$$\Leftrightarrow \min_{x^i \in \mathcal{R}^{n_i}} \{f^i(x^i; x^{-i}) : x^i = (z^i, y^i) \in \mathfrak{F}_i, 0 \le w^i_j \le Lv_j \ \forall j = 1, \dots, k,$$

$$0 \le \{M'w^i + q\}_j \le (1 - v_j)L \ \forall j = 1, \dots, k, v \in \{0,1\}^k\}$$

$L$ sufficiently large.

## Theorem ([Balas, 1985])

*Given $k$ polyhedra $S_i = \{x \in \mathcal{R}^n : A^i x \leq b^i\}$ for $i = 1, \ldots k$, then cl conv$(\bigcup_{i=1}^k S_i)$ is given by the set $\{x \in \mathcal{R}^n : \exists (x^1, \ldots, x^k, \delta) \in (\mathcal{R}^n)^k \times \mathcal{R}^k : x \in \{A^i x^i \leq \delta_i b^i, \sum_{w=1}^k x^w = x, \sum_{w=1}^k \delta_w = 1, \delta_i \geq 0, \forall i \in [k]\}\}$*

Idea 3: Leader $i$ mixed strategy belongs to the convex hull closure of their feasible set.

$$(P^i) \min_{w^i} \{f^i(x^i; x^{-i}) : w^i = (\overbrace{((z^i, y^i)}^{x^i}, \lambda^i), x^i \in \mathcal{F}_i, 0 \leq w^i_j \leq L v_j \ \forall j = 1, \ldots, k,$$

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Mixed strategy: $w^i = \sum_j \eta_j \hat{w}^i_j$ with $\hat{w}^i_j \in S^i_j \cap \mathcal{F}_i$ and $\sum_j \eta_j = 1$.

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$$\Leftrightarrow \min_{w^i}\{f^i(x^i; x^{-i}) : x^i \in \mathcal{F}_i, w^i = (x^i, \lambda^i) \in \bigcup_{j=1}^{2^k} S^i_j\}$$

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$$\Leftrightarrow \min_{w^i, \eta} \{\sum_j \eta_j f^i(x^i_j; x^{-i}) : x^i_j \in \mathcal{F}_i, w^i_j \in S^i_j, \sum_j \eta_j = 1\} \quad \textit{since the objective is linear}$$

$$\Leftrightarrow \min_{w^i} \{f^i(x^i; x^{-i}) : w^i = (\overbrace{(z^i, y^i)}^{x^i}, \lambda^i) \in \text{cl conv}(\bigcup_{j=1}^{k'} \left(S^i_j \cap \mathcal{F}_i\right))\}$$

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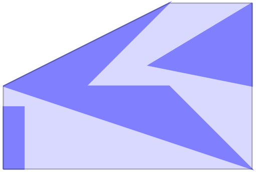
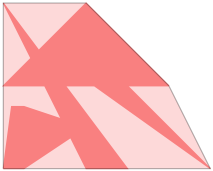
# Enumeration algorithm

Step 1: enumerate all polyhedra for each leader

# Enumeration algorithm

Step 2: compute the convex-hull of each leader





# Enumeration algorithm

Step 3: the leaders' game is equivalent to an LCP (which can be converted in a MIP)

# Enumeration algorithm

Step 4: the solution can be interpreted as a mixed strategy

# Inner approximation algorithm

Challenge: There can be exponentially many polyhedra!

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$$S = \left\{ x : \begin{array}{c} Ax \leq b \\ z = Mx + q \\ 0 \leq x_i \perp z_i \geq 0, \quad \forall\, i \in \mathcal{C} \end{array} \right\} = \bigcup_{j=1}^{2^{|\mathcal{C}|}} S_j$$

$$\operatorname{cl}\operatorname{conv}(S) \subseteq \mathcal{O}_0 = \{x : Ax \leq b,\ z = Mx + q,\ x_i \geq 0,\ z_i \geq 0 \ \ \forall\, i \in \mathcal{C}\}$$

# Inner approximation algorithm

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$$\text{cl conv}(S) \subseteq \mathcal{O}_0 = \{x : Ax \leq b,\ z = Mx + q,\ x_i \geq 0,\ z_i \geq 0\ \ \forall\, i \in \mathcal{C}\}$$

$$\text{cl conv}\left( \bigcup_{b \in J} \mathcal{P}(b) \cap \mathcal{O}_0 \right) \subseteq \text{cl conv}(S)$$

where
$\mathcal{P}(b) = \{x_{c_i} \leq 0,\ \forall\, i \in \{i : b_i = 0\}\} \bigcap \{[Mx + q]_{c_i} \leq 0,\ \forall\, i \in \{i : b_i = 1\}\}$

# Inner approximation algorithm



1. Construct an initial inner approximation $\hat{\mathcal{F}}^i$ of each leader $i$ feasible region
2. Solve the Nash game for the feasible strategies $\hat{\mathcal{F}}^i$
3. If step 2 found an equilibrium, verify if a player has incentive to deviate: if not, return equilibrium; otherwise go to step 4.

# Inner approximation algorithm

1. Construct an initial inner approximation $\hat{\mathcal{F}}^i$ of each leader $i$ feasible region
2. Solve the Nash game for the feasible strategies $\hat{\mathcal{F}}^i$
3. If step 2 found an equilibrium, verify if a player has incentive to deviate: if not, return equilibrium; otherwise go to step 4.
   Otherwise, for each player $i$, add a new set of polyhedra to $\hat{\mathcal{F}}^i$ and go to step 2.
4. Add the polyhedra corresponding to a player deviation. Go to step 2.

# Results

Instances with 3 to 5 leaders, and 3 followers.

| | Algorithm | ES | $k$ | Time (s) EQ | Time (s) NO | Time (s) All | Wins EQ | Wins NO | Solved |
|---|---|---|---|---|---|---|---|---|---|
| | *FE* | - | - | 26.78 | 0.12 | 120.21 | 6 | 82 | 140/149 |
| *MNE* | *InnerApp* | Seq | 1 | 6.18 | 0.35 | 51.33 | 3 | 0 | 145/149 |
| | | Seq | 3 | 16.20 | 0.18 | 55.82 | 5 | 0 | 145/149 |
| | | Seq | 5 | 5.85 | 0.15 | 51.08 | 3 | 0 | 145/149 |
| | | RSeq | 1 | 7.33 | 0.36 | 3.73 | 26 | 0 | 149/149 |
| | | RSeq | 3 | 10.31 | 0.18 | 53.12 | 4 | 0 | 145/149 |
| | | RSeq | 5 | 8.68 | 0.15 | 76.41 | 5 | 0 | 143/149 |
| | | Rand | 1 | 4.80 | 0.36 | 26.60 | 8 | 0 | 147/149 |
| | | Rand | 3 | 29.49 | 0.18 | 85.65 | 5 | 0 | 143/149 |
| | | Rand | 5 | 21.59 | 0.15 | 58.26 | 2 | 0 | 145/149 |
| *PNE* | *FE-P* | - | - | 6.46 | 0.12 | 328.23 | – | – | 122/149 |

# Results

Instances with 7 leaders, and up to 3 followers.

| | Algorithm | ES | $k$ | Time (s) EQ | Time (s) NO | All | Wins EQ | Wins NO | Solved |
|---|---|---|---|---|---|---|---|---|---|
| | *FE* | - | - | 260.29 | 1.12 | 1174.32 | 0 | 2 | 20/50 |
| *MNE* | *InnerApp* | Seq | 1 | 39.26 | 9.64 | 672.24 | 1 | 0 | 32/50 |
| | | Seq | 3 | 62.66 | 3.88 | 616.25 | 1 | 0 | 34/50 |
| | | Seq | 5 | 24.03 | 2.83 | 733.97 | 1 | 0 | 30/50 |
| | | Rev.Seq | 1 | 171.47 | 9.66 | 262.74 | 27 | 0 | 47/50 |
| | | Rev.Seq | 3 | 13.85 | 3.86 | 585.27 | 4 | 0 | 34/50 |
| | | Rev.Seq | 5 | 78.57 | 2.83 | 798.90 | 6 | 0 | 29/50 |
| | | Random | 1 | 34.65 | 9.65 | 497.06 | 0 | 0 | 37/50 |
| | | Random | 3 | 123.02 | 3.87 | 588.03 | 2 | 0 | 36/50 |
| | | Random | 5 | 39.18 | 2.86 | 711.77 | 4 | 0 | 41/50 |
| *PNE* | *FE-P* | - | - | 7.36 | 1.12 | 1441.95 | – | – | 10/50 |

# Contents

## Reciprocally-Bilinear Games [Carvalho et al., 2021b]

A Reciprocally-Bilinear Game (RBG) is a game among a finite set of players $N$ such that the utility and the strategy set of player $i \in N$ is as follows:

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We saw NASPs which are polyhedrally-representable RBGs.

$$\max_{x^i} \ (c^i)^T x^i + (x^{-i})^T C^i x^i$$
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### Theorem

*Given an RBG $G$ and a copy of it $\tilde{G}$ where the feasible region of player $i$ is $clconv(\mathcal{X}^i)$ (instead of $\mathcal{X}^i$) then*

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**Theorem**

*Given an RBG $G$ and a copy of it $\tilde{G}$ where the feasible region of player $i$ is $clconv(\mathcal{X}^i)$ (instead of $\mathcal{X}^i$) then*

1. *For any pure equilibrium $\tilde{\sigma}$ of $\tilde{G}$, there is an equilibrium $\sigma$ of $G$.*
2. *If $\tilde{G}$ has no pure equilibrium, then $G$ has no equilibrium.*

$$\max_{x^i} \; (c^i)^T x^i + (x^{-i})^T C^i x^i$$
$$s.t. \; x^i \in \mathcal{X}^i.$$

If $A^i$ and $b^i$ describe $clconv(\mathcal{X}^i)$

Computing equilibria of RBG $G \equiv$
Computing **pure** equilibria of RBG $\tilde{G}$

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If $A^i$ and $b^i$ **DO NOT** describe $clconv(\mathcal{X}^i)$

Idea: Iteratively improve outer approximations of $clconv(\mathcal{X}^i)$.

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If $A^i$ and $b^i$ **DO NOT** describe $clconv(\mathcal{X}^i)$

Idea: Iteratively improve outer approximations of $clconv(\mathcal{X}^i)$.

Either we find an equilibrium for $G$ or we refine the approximation in $\tilde{G}$

$$\max_{x^i} \; (c^i)^T x^i + (x^{-i})^T C^i x^i$$
$$s.t. \; x^i \in \mathcal{X}^i.$$

Given the polyhedrally-representable RBG $G$, we construct polyhedral approximate game $\tilde{G}$ where each solves instead:

$$\max_{x^i} \; (c^i)^T x^i + (x^{-i})^T C^i x^i$$
$$s.t. \; x^i \in \tilde{\mathcal{X}}^i.$$

$$\tilde{\mathcal{X}}^i = \{x^i : \tilde{A}^i x^i \leq \tilde{b}^i, x^i \geq 0\}$$
$$\mathcal{X}^i \subseteq clconv(\mathcal{X}^i) \subseteq \tilde{\mathcal{X}}^i$$

Outer approximation:

$$\max_{x^i} \; (c^i)^T x^i + (x^{-i})^T C^i x^i$$
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We use LCPs to find a pure equilibrium for $\tilde{G}$.

1. Given an RBG $G$, we determine a “convexified” outer approximation RBG $\tilde{G}$

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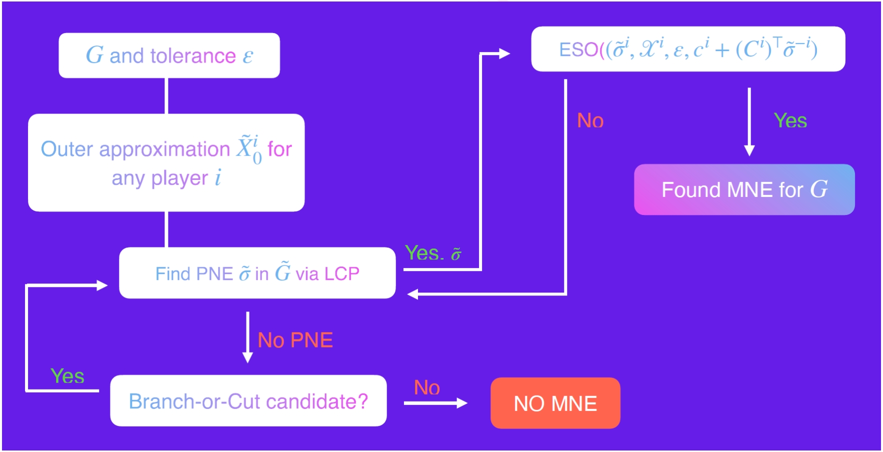
4. If no $\tilde{\sigma}$ in step 2, branch or cut.

# Cut-and-Play




Code: https://docs.getzero.one/

# Contents

Summary:

- Mathematical programming games encompass flexible problem modeling.
- Although the theoretical intractability of NASPs, in practice, we can efficiently compute equilibria.
- Tools to compute Nash equilibria allow us to understand the benefits and issues of the games.

Future work & recent literature:

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- New solution concepts.

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